



## On the slice genus of links

VINCENT FLORENS  
PATRICK M. GILMER

**Abstract** We define Casson-Gordon  $\sigma$ -invariants for links and give a lower bound of the slice genus of a link in terms of these invariants. We study as an example a family of two component links of genus  $h$  and show that their slice genus is  $h$ , whereas the Murasugi-Tristram inequality does not obstruct this link from bounding an annulus in the 4-ball.

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## 1 Introduction

A knot in  $S^3$  is slice if it bounds a smooth 2-disk in the 4-ball  $B^4$ . Levine showed [Le] that a slice knot is algebraically slice, i.e. any Seifert form of a slice knot is metabolic. In this case, the Tristram-Levine signatures at the prime power order roots of unity of a slice knot must be zero. Levine showed also that the converse holds in high odd dimensions, i.e. any algebraically slice knot is slice. This is false in dimension 3: Casson and Gordon [CG1, CG2, G] showed that certain two-bridge knots in  $S^3$ , which are algebraically slice, are not slice knots. For this purpose, they defined several knot and 3-manifold invariants, closely related to the Tristram-Levine signatures of associated links. Further methods to calculate these invariants were developed by Gilmer [Gi3, Gi4], Litherland [Li], Gilmer-Livingston [GL], and Naik [N]. Lines [L] also computed some of these invariants for some fibered knots, which are algebraically slice but not slice. The slice genus of a link is the minimal genus for a smooth oriented connected surface properly embedded in  $B^4$  with boundary the given link.

The Murasugi-Tristram inequality (see Theorem 2.1 below) gives a lower bound on the slice genus of a link in terms of the link's Tristram-Levine signatures and related nullity invariants. The second author [Gi1] used Casson-Gordon invariants to give another lower bound on the slice genus of a knot. In particular

he gave examples of algebraically slice knots whose slice genus is arbitrarily large. We apply these methods to restrict the slice genus of a link.

We study as an example a family of two component links, which have genus  $h$  Seifert surfaces. Using Theorem 4.1, we show that these links cannot bound a smoothly embedded surface in  $B^4$  with genus lower than  $h$ , while the Murasugi-Tristram inequality does not show this. In fact there are some links with the same Seifert form that bound annuli in  $B^4$ . We work in the smooth category.

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## 2 Preliminaries

### 2.1 The Tristram-Levine signatures

Let  $L$  be an oriented link in  $S^3$ , with  $\mu$  components, and  $\theta_S$  be the Seifert pairing corresponding to a connected Seifert surface  $S$  of the link. For any complex number  $\lambda$  with  $|\lambda| = 1$ , one considers the hermitian form  $\theta_S^\lambda := (1 - \lambda)\theta_S + (1 - \bar{\lambda})(\theta_S)^T$ . The Tristram signature  $\sigma_L(\lambda)$  and nullity  $n_L(\lambda)$  of  $L$  are defined as the signature and nullity of  $\theta_S^\lambda$ . Levine defined these same signatures for knots [Le]. The Alexander polynomial of  $L$  is  $\Delta_L(t) := \text{Det}(\theta_S - t(\theta_S)^T)$ . As is well-known,  $\sigma_L$  is a locally constant map on the complement in  $S^1$  of the roots of  $\Delta_L$  and  $n_L$  is zero on this complement. If  $\Delta_L = 0$ , it is still true that the signature and nullity are locally constant functions on the complement of some finite collection of points.

The Murasugi-Tristram inequality allows one to estimate the slice genus of  $L$ , in terms of the values of  $\sigma_L(\lambda)$  and  $n_L(\lambda)$ .

**Theorem 2.1** [M, T] *Suppose that  $L$  is the boundary of a properly embedded connected oriented surface  $F$  of genus  $g$  in  $B^4$ . Then, if  $\lambda$  is a prime power order root of unity, we have*

$$|\sigma_L(\lambda)| + n_L(\lambda) \leq 2g + \mu - 1.$$

### 2.2 The Casson-Gordon $\sigma$ -invariant

In this section, for the reader convenience, we review the definition and some of the properties of the simplest kind of Casson-Gordon invariant. It is a reformulation of the Atiyah-Singer  $\alpha$ -invariant.

Let  $M$  be an oriented compact three manifold and  $\chi: H_1(M) \rightarrow \mathbb{C}^*$  be a character of finite order. For some  $q \in \mathbb{N}^*$ , the image of  $\chi$  is contained a cyclic subgroup of order  $q$  generated by  $\alpha = e^{2i\pi/q}$ . As  $\text{Hom}(H_1(M), C_q) = [M, B(C_q)]$ , it follows that  $\chi$  induces  $q$ -fold covering of  $M$ , denoted  $\widetilde{M}$ , with a canonical deck transformation. We will denote this transformation also by  $\alpha$ . If  $\chi$  maps onto  $C_q$ , the canonical deck transformation sends  $x$  to the other endpoint of the arc

that begins at  $x$  and covers a loop representing an element of  $(\chi)^{-1}(\alpha)$ .

As the bordism group  $\Omega_3(B(C_q)) = C_q$ , we may conclude that  $n$  disjoint copies of  $M$ , for some integer  $n$ , bounds bound a compact 4-manifold  $W$  over  $B(C_q)$ . Note  $n$  can be taken to be  $q$ . Let  $\widetilde{W}$  be the induced covering with the deck transformation, denoted also by  $\alpha$ , that restricts to  $\alpha$  on the boundary. This induces a  $\mathbb{Z}[C_q]$ -module structure on  $C_*(\widetilde{W})$ , where the multiplication by  $\alpha \in \mathbb{Z}[C_q]$  corresponds to the action of  $\alpha$  on  $\widetilde{W}$ .

The cyclotomic field  $\mathbb{Q}(C_q)$  is a natural  $\mathbb{Z}[C_q]$ -module and the twisted homology  $H_*^t(W; \mathbb{Q}(C_q))$  is defined as the homology of

$$C_*(\widetilde{W}) \otimes_{\mathbb{Z}[C_q]} \mathbb{Q}(C_q).$$

Since  $\mathbb{Q}(C_q)$  is flat over  $\mathbb{Z}[C_q]$ , we get an isomorphism

$$H_*^t(W; \mathbb{Q}(C_q)) \simeq H_*(\widetilde{W}) \otimes_{\mathbb{Z}[C_q]} \mathbb{Q}(C_q).$$

Similarly, the twisted homology  $H_*^t(M; \mathbb{Q}(C_q))$  is defined as the homology of

$$C_*(\widetilde{M}) \otimes_{\mathbb{Z}[C_q]} \mathbb{Q}(C_q).$$

Let  $\widetilde{\phi}$  be the intersection form on  $H_2(\widetilde{W}; \mathbb{Q})$  and define

$$\phi_\chi(W): H_2^t(W; \mathbb{Q}(C_q)) \times H_2^t(W; \mathbb{Q}(C_q)) \rightarrow \mathbb{Q}(C_q)$$

so that, for all  $a, b$  in  $\mathbb{Q}(C_q)$  and  $x, y$  in  $H_2(\widetilde{W})$ ,

$$\phi_\chi(W)(x \otimes a, y \otimes b) = \bar{a}\bar{b} \sum_{i=1}^q \widetilde{\phi}(x, \alpha^i y) \bar{\alpha}^i,$$

where  $a \rightarrow \bar{a}$  denotes the involution on  $\mathbb{Q}(C_q)$  induced by complex conjugation.

**Definition 2.2** The Casson-Gordon  $\sigma$ -invariant of  $(M, \chi)$  and the related nullity are

$$\begin{aligned} \sigma(M, \chi) &:= \frac{1}{n} (\text{Sign}(\phi_\chi(W)) - \text{Sign}(W)) \\ \eta(M, \chi) &:= \dim H_1^t(M; \mathbb{Q}(C_q)). \end{aligned}$$

If  $U$  is a closed 4-manifold and  $\chi: H_1(U) \rightarrow C_q$  we may define  $\phi_\chi(U)$  as above. One has that modulo torsion the bordism group  $\Omega_4(B(C_q))$  is generated by the constant map from  $CP(2)$  to  $B(C_q)$ . If  $\chi$  is trivial, one has that  $\text{Sign}(\phi_\chi(U)) = \text{Sign}(U)$ . Since both signatures are invariant under cobordism, one has in general that  $\text{Sign}(\phi_\chi(U)) = \text{Sign}(U)$ . The independence of  $\sigma(M, \chi)$  from the choice of  $W$  and  $n$  follows from this and Novikov additivity. One may see directly that these invariants do not depend on the choice of  $q$ . In this way Casson and Gordon argued that  $\sigma(M, \chi)$  is an invariant. Alternatively one may use the Atiyah-Singer G-Signature theorem and Novikov additivity [AS].

We now describe a way to compute  $\sigma(M, \chi)$  for a given surgery presentation of  $(M, \chi)$ .

**Definition 2.3** Let  $K$  be an oriented knot in  $S^3$ . Let  $A$  be an embedded annulus such that  $\partial A = K \cup K'$  with  $lk(K, K') = f$ . A  $p$ -cable on  $K$  with twist  $f$  is defined to be the union of oriented parallel copies of  $K$  lying in  $A$  such that the number of copies with the same orientation minus the number with opposite orientation is equal to  $p$ .

Let us suppose that  $M$  is obtained by surgery on a framed link  $L = L_1 \cup \dots \cup L_\mu$  with framings  $f_1, \dots, f_\mu$ . One shows that the linking matrix  $\Lambda$  of  $L$  with framings in the diagonal is a presentation matrix of  $H_1(M)$  and a character on  $H_1(M)$  is determined by  $\alpha^{p_i} = \chi(m_{L_i}) \in C_q$  where  $m_{L_i}$  denotes the class of the meridian of  $L_i$ . Let  $\vec{p} = (p_1, \dots, p_\mu)$ . We use the following generalization of a formula in [CG2, Lemma (3.1)], where all  $p_i$  are assumed to be 1, that is given in [Gi2, Theorem(3.6)].

**Proposition 2.4** Suppose  $\chi$  maps onto  $C_q$ . Let  $L'$  with  $\mu'$  components be the link obtained from  $L$  by replacing each component by a non-empty algebraic  $p_i$ -cable with twist  $f_i$  along this component. Then, if  $\lambda = e^{2ir\pi/q}$ , for  $(r, q) = 1$ , one has

$$\sigma(M, \chi^r) = \sigma_{L'}(\lambda) - \text{Sign}(\Lambda) + 2 \frac{r(q-r)}{q^2} \vec{p}^\top \Lambda \vec{p},$$

$$\eta(M, \chi^r) = \eta_{L'}(\lambda) - \mu' + \mu.$$

The following proposition collects some easy additivity properties of the  $\sigma$ -invariant and the nullity under the connected sum.

**Proposition 2.5** Suppose that  $M_1, M_2$  are connected. Then,

for all  $\chi_i \in H^1(M_i; C_q)$ ,  $i = 1, 2$ , we have

$$\sigma(M_1 \# M_2, \chi_1 \oplus \chi_2) = \sigma(M_1, \chi_1) + \sigma(M_2, \chi_2).$$

If both  $\chi_i$  are non-trivial, then

$$\eta(M_1 \# M_2, \chi_1 \oplus \chi_2) = \eta(M_1, \chi_1) + \eta(M_2, \chi_2) + 1.$$

If one  $\chi_i$  is trivial, then

$$\eta(M_1 \# M_2, \chi_1 \oplus \chi_2) = \eta(M_1, \chi_1) + \eta(M_2, \chi_2).$$

**Proposition 2.6** For all  $\chi \in H_1(S^1 \times S^2; C_q)$ , we have

$$\sigma(S^1 \times S^2, \chi) = 0$$

If  $\chi \neq 0$ , then  $\eta(S^1 \times S^2, \chi) = 0$ . If  $\chi = 0$ , then  $\eta(S^1 \times S^2, \chi) = 1$ .

Proposition 2.6 for non-trivial  $\chi$  can be proved for example by the use of Proposition 2.4, since  $S^1 \times S^2$  is obtained by surgery on the unknot framed 0. However it is simplest to derive this result directly from the definitions.

### 2.3 The Casson-Gordon $\tau$ -invariant

In this section, we recall the definition and some of the properties of the Casson-Gordon  $\tau$ -invariant. Let  $C_\infty$  denote a multiplicative infinite cyclic group generated by  $t$ . For  $\chi^+ : H_1(M) \rightarrow C_q \oplus C_\infty$ , we denote  $\bar{\chi} : H_1(M) \rightarrow C_q$  the character obtained by composing  $\chi^+$  with projection on the first factor. The character  $\chi^+$  induces a  $C_q \times C_\infty$ -covering  $\widetilde{M}_\infty$  of  $M$ .

Since the bordism group  $\Omega_3(B(C_q \times C_\infty)) = C_q$ , bounds a compact 4-manifold  $W$  over  $B(C_q \times C_\infty)$ . Again  $n$  can be taken from to be  $q$ .

If we identify  $\mathbb{Z}[C_q \times C_\infty]$  with the Laurent polynomial ring  $\mathbb{Z}[C_q][t, t^{-1}]$ , the field  $\mathbb{Q}(C_q)(t)$  of rational functions over the cyclotomic field  $\mathbb{Q}(C_q)$  is a flat  $\mathbb{Z}[C_q \times C_\infty]$ -module. We consider the chain complex  $C_*(\widetilde{W}_\infty)$  as a  $\mathbb{Z}[C_q \times C_\infty]$ -module given by the deck transformation of the covering. Since  $W$  is compact, the vector space  $H_2^t(W; \mathbb{Q}(C_q)(t)) \simeq H_2(\widetilde{W}_\infty) \otimes_{\mathbb{Z}[C_q][t, t^{-1}]} \mathbb{Q}(C_q)(t)$  is finite dimensional.

We let  $J$  denote the involution on  $\mathbb{Q}(C_q)(t)$  that is linear over  $\mathbb{Q}$  sends  $t^i$  to  $t^{-i}$  and  $\alpha^i$  to  $\alpha^{-i}$ . As in [G], one defines a hermitian form, with respect to  $J$ ,

$$\phi_{\chi^+} : H_2^t(W; \mathbb{Q}(C_q)(t)) \times H_2^t(W; \mathbb{Q}(C_q)(t)) \rightarrow \mathbb{Q}(C_q)(t),$$

such that

$$\phi_{\chi^+}(x \otimes a, y \otimes b) = J(a) \cdot b \cdot \sum_{i \in \mathbb{Z}} \sum_{j=1}^q \widetilde{\phi^+}(x, t^i \alpha^j y) \bar{\alpha}^j t^{-i}.$$

Here  $\widetilde{\phi^+}$  denotes the ordinary intersection form on  $\widetilde{W}_\infty$ . Let  $\mathcal{W}(\mathbb{Q}(C_q)(t))$  be the Witt group of non-singular hermitian forms on finite dimensional  $\mathbb{Q}(C_q)(t)$  vector spaces. Let us consider  $H_2^t(W; \mathbb{Q}(C_q)(t)) / (\text{Radical}(\phi_{\chi^+}))$ . The induced form on it represents an element in  $\mathcal{W}(\mathbb{Q}(C_q)(t))$ , which we denote  $w(W)$ . Furthermore, the ordinary intersection form on  $H_2(W; \mathbb{Q})$  represents an element of  $\mathcal{W}(\mathbb{Q})$ . Let  $w_0(W)$  be the image of this element in  $\mathcal{W}(\mathbb{Q}(C_q)(t))$ .

**Definition 2.7** The Casson-Gordon  $\tau$ -invariant of  $(M, \chi^+)$  is

$$\tau(M, \chi^+) := \frac{1}{n} (w(W) - w_0(W)) \in \mathcal{W}(\mathbb{Q}(C_q)(t)) \otimes \mathbb{Q}.$$

Suppose that  $nM$  bounds another compact 4-manifold  $W'$  over  $B(C_q \times C_\infty)$ . Form the closed compact manifold over  $B(C_q \times C_\infty)$ ,  $U := W \cup W'$  by gluing along the boundary. By Novikov additivity, we get  $w(U) - w_0(U) = (w(W) - w_0(W)) - (w(W') - w_0(W'))$ . Using [CF], the bordism group  $\Omega_4(B(C_q \times C_\infty))$ , modulo torsion, is generated by  $CP(2)$ , with the constant map to  $B(C_q \times C_\infty)$ . We have that  $w(CP(2)) = w_0(CP(2))$ . Since  $w(U)$ , and  $w_0(U)$  only depend on the bordism class of  $U$  over  $B(C_q \times C_\infty)$ , it follows that  $w(U) = w_0(U)$  and  $\tau(M, \chi^+)$  is independent of the choice of  $W$ . Using the above techniques, one may check  $\tau(M, \chi^+)$  is independent of  $n$ .

If  $A \in \mathcal{W}(\mathbb{Q}(C_q)(t))$ , let  $A(t)$  be a matrix representative for  $A$ . The entries of  $A(t)$  are Laurent polynomials with coefficients in  $\mathbb{Q}(C_q)$ . If  $\lambda$  is in  $S^1 \subset \mathbb{C}$ , then  $A(\lambda)$  is hermitian and has a well defined signature  $\sigma_\lambda(A)$ . One can view  $\sigma_\lambda(A)$  as a locally constant map on the complement of the set of the zeros of  $\det A(\lambda)$ . As in [CG1], we re-define  $\sigma_\lambda(A)$  at each point of discontinuity as the average of the one-sided limits at the point.

We have the following estimate [Gi3, Equation (3.1)].

**Proposition 2.8** Let  $\chi^+ : H_1(M) \rightarrow C_q \oplus C_\infty$  and  $\bar{\chi} : H_1(M) \rightarrow C_q$  be  $\chi^+$  followed by the projection to  $C_q$ . We have

$$|\sigma_1(\tau(M, \chi^+)) - \sigma(M, \bar{\chi})| \leq \eta(M, \bar{\chi}).$$

## 2.4 Linking forms

Let  $M$  be a rational homology 3-sphere with linking form

$$l: H_1(M) \times H_1(M) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

We have that  $l$  is non-singular, that is the adjoint of  $l$  is an isomorphism  $\iota: H_1(M) \rightarrow \text{Hom}(H_1(M), \mathbb{Q}/\mathbb{Z})$ . Let  $H_1(M)^*$  denote  $\text{Hom}(H_1(M), \mathbb{C}^*)$ . Let  $\nu$  denote the map  $\mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{C}^*$  that sends  $\frac{a}{b}$  to  $e^{\frac{2\pi ia}{b}}$ . So we have an isomorphism  $j: H_1(M) \rightarrow H_1(M)^*$  given by  $x \mapsto \nu \circ \iota(x)$ . Let  $\beta: H_1(M)^* \times H_1(M)^* \rightarrow \mathbb{Q}/\mathbb{Z}$  be the dual form defined by  $\beta(jx, jy) = -l(x, y)$ .

**Definition 2.9** The form  $\beta$  is metabolic with metabolizer  $H$  if there exists a subgroup  $H$  of  $H_1(M)^*$  such that  $H^\perp = H$ .

**Lemma 2.10** [G1] *If  $M$  bounds a spin 4-manifold  $W$  then  $\beta = \beta_1 \oplus \beta_2$  where  $\beta_2$  is metabolic and  $\beta_1$  has an even presentation with rank  $\dim H_2(W; \mathbb{Q})$  and signature  $\text{Sign}(W)$ . Moreover, the set of characters that extend to  $H_1(W)$  forms a metabolizer for  $\beta_2$ .*

## 2.5 Link invariants

Let  $L = L_1 \cup \cdots \cup L_\mu$  be an oriented link in  $S^3$ . Let  $N_2$  be the two-fold covering of  $S^3$  branched along  $L$  and  $\beta_L$  be the linking form on  $H_1(N_2)^*$ , see previous section.

We suppose that the Alexander polynomial of  $L$  satisfies

$$\Delta_L(-1) \neq 0.$$

Hence,  $N_2$  is a rational homology sphere. Note that if  $\Delta_L(-1) \neq 1$ , then  $H_1(N_2; \mathbb{Z})$  is non-trivial.

**Definition 2.11** For all characters  $\chi$  in  $H_1(N_2)^*$ , the Casson-Gordon  $\sigma$ -invariant of  $L$  and the related nullity are (see Definition 2.2):

$$\begin{aligned}\sigma(L, \chi) &:= \sigma(N_2, \chi), \\ \eta(L, \chi) &:= \eta(N_2, \chi).\end{aligned}$$

**Remark 2.12** If  $L$  is a knot, then Definition 2.11 coincides with  $\sigma(L, \chi)$  defined in [CG1, p.183].

### 3 Framed link descriptions

In this section, we study the Casson-Gordon  $\tau$ -invariants of the two-fold cover  $M_2$  of the manifold  $M_0$  described below.

Let  $S^3 - T(L)$  be the complement in  $S^3$  of an open tubular neighborhood of  $L$  in  $S^3$  and  $P$  be a planar surface with  $\mu$  boundary components.

Let  $S$  be a Seifert surface for  $L$  and  $\gamma_i$  for  $i = 1, \dots, \mu$  be the curves where  $S$  intersects the boundary of  $S^3 - T(L)$ . We define  $M_0$  as the result of gluing  $P \times S^1$  to  $S^3 - T(L)$ , where  $P \times 1$  is glued along the curves  $\gamma_i$ . Let  $*$  be a point in the boundary of  $P$ .

A recipe for drawing a framed link description for  $M_0$  is given in the proof of Proposition 3.1.

**Proposition 3.1**

$$H_1(M_0) \simeq \mathbb{Z} \oplus \mathbb{Z}^{\mu-1} \simeq \langle m \rangle \oplus \mathbb{Z}^{\mu-1},$$

where  $m$  denotes the class of  $* \times S^1$  in  $P \times S^1$ .

**Proof** Form a 4-manifold  $X$  by gluing  $P \times D^2$  to  $D^4$  along  $S^3$  in such a way that the total framing on  $L$  agrees with the Seifert surface  $S$ . The boundary of this 4-manifold is  $M_0$ . We can get a surgery description of  $M_0$  in the following way: pick  $\mu - 1$  paths of  $S$  joining up the components of  $L$  in a chain. Deleting open neighborhoods of these paths in  $S$  gives a Seifert surface for a knot  $L'$  obtained by doing a fusion of  $L$  along bands that are neighborhoods of the original paths. Put a circle with a dot around each of these bands (representing a 4-dimensional 1-handle in Kirby's [K] notation), and the framing zero on  $L'$ . This describes a handlebody decomposition of  $X$ .

One can then get a standard framed link description of  $M_0$  by replacing the circle with dots with unknots  $T_1, \dots, T_{\mu-1}$  framed zero. This changes the 4-manifold but not the boundary. Note also that  $lk(T_i, T_j) = 0$  and  $lk(T_i, L') = 0$  for all  $i = 1, \dots, \mu - 1$ . Hence  $H_1(M_0) \simeq \mathbb{Z}^\mu$  and  $m$  represents one of the generators.  $\square$

We now consider an infinite cyclic covering  $M_\infty$  of  $M_0$ , defined by a character  $H_1(M_0) \rightarrow C_\infty = \langle t \rangle$  that sends  $m$  to  $t$  and the other generators to zero. Let us denote by  $M_2$  the intermediate two-fold covering obtained by composing this character with the quotient map  $C_\infty \rightarrow C_2$  sending  $t$  to  $-1$ . Let  $m_2$  denote the loop in  $M_2$  given by the inverse image of  $m$ . A recipe for drawing a framed link description for  $M_2$  is given in the proof of Remark 3.3.



**Proposition 3.2** *There is an isomorphism between  $H_1(N_2)$  and the torsion subgroup of  $H_1(M_2)$ , which only depends on  $L$ . Moreover*

$$H_1(M_2) \simeq H_1(N_2) \oplus \mathbb{Z}^\mu \simeq H_1(N_2) \oplus \langle m_2 \rangle \oplus \mathbb{Z}^{\mu-1}.$$

**Proof** Let  $R$  be the result of gluing  $P \times D^2$  to  $S^3 \times I$  along  $L \times 1 \subset S^3 \times 1$  using the framing given by the Seifert surface. Thus  $R$  is the result of adding  $\mu - 1$  1-handles to  $S^3 \times I$  and then one 2-handle along  $L'$ , as in the proof above. Then  $X$  in the proof above can be obtained by gluing  $D^4$  to  $R$  along  $S^3 \times 0$ . Since  $D^2$  is the double branched cover of itself along the origin,  $P \times D^2$  is the double branched cover of itself along  $P \times 0$ . Let  $R_2$  denote the double branched cover of  $R$  that is obtained by gluing  $P \times D^2$  to  $N_2 \times I$  along a neighborhood of the lift of  $L \times 1 \subset S^3 \times 1$ . We have that  $\partial R_2 = -N_2 \sqcup M_2$ , where  $R_2$  is the result of adding  $\mu - 1$  1-handles to  $N_2 \times I$  and then one 2-handle along the lift  $L'$ . Moreover this lift of  $L'$  is null-homologous in  $N_2$ . It follows that  $H_1(R_2)$  is isomorphic to  $H_1(N_2) \oplus \mathbb{Z}^{\mu-1}$ , with the inclusion of  $N_2$  into  $R_2$  inducing an isomorphism  $i_N$  of  $H_1(N_2)$  to the torsion subgroup of  $H_1(R_2)$ . Turning this handle decomposition upside down we have that  $R_2$  is the result of adding to  $M_2 \times I$  one 2-handle along a neighborhood of  $m_2$  and then  $\mu - 1$  3-handles. It follows that  $H_1(R_2) \oplus \mathbb{Z} = H_1(R_2) \oplus \langle m_2 \rangle$  is isomorphic to  $H_1(M_2)$  with the inclusion of  $M_2$  in  $R_2$  inducing an isomorphism  $i_M$  of the torsion subgroup  $H_1(M_2)$  to the torsion subgroup of  $H_1(R_2)$ . Thus  $(i_M)^{-1} \circ i_N$  is an isomorphism from  $H_1(N_2)$  to the torsion subgroup of  $H_1(M_2)$  and this isomorphism is constructed without any arbitrary choices.  $\square$

**Remark 3.3** We could have proved Proposition 3.1 in a similar way to the proof of Proposition 3.2. We could have also proved Proposition 3.2 (except for the isomorphism only depending on  $L$ ) in a similar way to the proof of Proposition 3.1 as follows. We can find a surgery description of  $M_2$  from a surgery description of  $N_2$ . The procedure of how to visualize a lift of  $L$  and the surface  $S$  in  $N_2$  is given in [AK]. One considers the lifts of the paths chosen in the proof of Proposition 3.1, on the lift of  $S$ . One then fuses the components of the lift of  $L$  along these paths, obtaining a lift of  $L'$ . The surgery description of  $M_2$  is obtained by adding to the surgery description of  $N_2$  the lift of  $L'$  with zero framing together with  $\mu - 1$  more unknotted zero-framed components encircling each fusion. The linking matrix of this link is a direct sum of that of  $N_2$  and a  $\mu \times \mu$  zero matrix.

Let  $i_T$  denote the inclusion of the torsion subgroup of  $H_1(M_2)$  into  $H_1(M_2)$ , and let  $\psi: H_1(N_2) \rightarrow H_1(M_2)$  denote the monomorphism given by  $i_T \circ (i_M)^{-1} \circ i_N$ .

**Theorem 3.4** *Let  $\chi^+: H_1(M_2) \rightarrow C_q \oplus C_\infty$ . Let  $\chi: H_1(N_2) \rightarrow C_q$  be  $\chi^+ \circ \psi$  composed with the projection to  $C_q$ . We have that:*

$$|\sigma_1(\tau(M_2, \chi^+)) - \sigma(L, \chi)| \leq \eta(L, \chi) + \mu.$$

**Remark 3.5** If  $L$  is a knot, then  $\tau(M_2, \chi^+)$  coincides with  $\tau(L, \chi)$  defined in [CG1, p.189].

**Proof of Theorem 3.4** We use the surgery description of  $M_2$  given in Remark 3.3. Let  $P$  be given by the surgery description of  $M_2$  but with the component corresponding to  $L'$  deleted. Hence,

$$P = N_2 \#_{(\mu-1)} S^1 \times S^2.$$

$\chi^+$  induces some character  $\chi'$  on  $H_1(P)$ .

According to Section 2.3, we let  $\bar{\chi} \in H^1(M_2; C_q)$  and  $\bar{\chi}' \in H^1(P; C_q)$  denote the characters  $\chi^+$  and  $\chi'$  followed by the projection  $C_q \oplus C_\infty \rightarrow C_q$ . Using Propositions 2.5 and 2.6, one has that

$$\sigma(P, \bar{\chi}') = \sigma(L, \chi) \text{ and } \eta(P, \bar{\chi}') = \eta(L, \chi) + \mu - 1.$$

Moreover, since  $M_2$  is obtained by surgery on  $L'$  in  $P$ , it follows from [Gi3, Proposition (3.3)] that

$$\begin{aligned} |\sigma(P, \bar{\chi}') - \sigma(M_2, \bar{\chi})| + |\eta(M_2, \bar{\chi}) - \eta(P, \bar{\chi}')| &\leq 1 \text{ or} \\ |\sigma(L, \chi) - \sigma(M_2, \bar{\chi})| + |\eta(M_2, \bar{\chi}) - \eta(L, \chi) - \mu + 1| &\leq 1. \end{aligned}$$

Thus

$$|\sigma(L, \chi) - \sigma(M_2, \bar{\chi})| \leq \eta(L, \chi) + \mu - \eta(M_2, \bar{\chi}).$$

Finally, one gets, by Theorem 2.8,

$$\begin{aligned} |\sigma_1(\tau(M_2, \chi^+)) - \sigma(L, \chi)| &\leq |\sigma_1(\tau(M_2, \chi^+)) - \sigma(M_2, \bar{\chi})| + |\sigma(M_2, \bar{\chi}) - \sigma(L, \chi)| \\ &\leq \eta(M_2, \bar{\chi}) + \eta(L, \chi) + \mu - \eta(M_2, \bar{\chi}) = \eta(L, \chi) + \mu. \quad \square \end{aligned}$$

## 4 The slice genus of links

See Section 2.5 for notations.

**Theorem 4.1** *Suppose  $L$  is the boundary of a connected oriented properly embedded surface  $F$  of genus  $g$  in  $B^4$ , and that  $\Delta_L(-1) \neq 0$ . Then,  $\beta_L$  can be written as a direct sum  $\beta_1 \oplus \beta_2$  such that the following two conditions hold:*

- 1)  $\beta_1$  has an even presentation of rank  $2g + \mu - 1$  and signature  $\sigma_L(-1)$ , and  $\beta_2$  is metabolic.
- 2) There is a metabolizer for  $\beta_2$  such that for all characters  $\chi$  of prime power order in this metabolizer,

$$|\sigma(L, \chi) + \sigma_L(-1)| \leq \eta(L, \chi) + 4g + 3\mu - 2.$$

**Proof** We let  $b_i(X)$  denote the  $i$ th Betti number of a space  $X$ . We have  $b_1(F) = 2g + \mu - 1$ .

Let  $W'_0$ , with boundary  $M'_0$ , be the complement of an open tubular neighborhood of  $F$  in  $B^4$ . By the Thom isomorphism, excision, and the long exact sequence of the pair  $(B^4, W'_0)$ ,  $W'_0$  has the homology of  $S^1$  wedge  $b_1(F)$  2-spheres. Let  $W'_2$  with boundary  $M'_2$  be the two-fold covering of  $W'_0$ . Note that if  $F$  is planar,  $M'_0 = M_0$ , and  $M'_2 = M_2$  (see Section 3).

Let  $V_2$  be the two-fold covering of  $B^4$  with branched set  $F$ . Note that  $V_2$  is spin as  $w_2(V_2)$  is the pull-up of a class in  $H^2(B^4, \mathbb{Z}_2)$ , by [Gi5, Theorem 7], for instance. The boundary of  $V_2$  is  $N_2$ . As in [Gi1], one calculates that  $b_2(V_2) = 2g + \mu - 1$ . One has  $\text{Sign}(V_2) = \sigma_L(-1)$  by [V].

By Lemma 2.10,  $\beta_L$  can be written as a direct sum  $\beta_1 \oplus \beta_2$  as in condition 1) above, such that the characters on  $H_1(N_2)$  that extend to  $H_1(V_2)$  form a metabolizer  $H$  for  $\beta_2$ . We now suppose  $\chi \in H$  and show that Condition 2) holds for  $\chi$ .

We also let  $\chi$  denote an extension of  $\chi$  to  $H_1(V_2)$  with image some cyclic group  $C_q$  where  $q$  is a power of a prime integer (possibly larger than those corresponding to the character on  $H_1(N_2)$ ). Of course  $\chi \in H^1(V_2, C_q)$  restricted to  $W'_2$  extends  $\chi$  restricted to  $M'_2$ . We simply denote all these restrictions by  $\chi$ .

Let  $W'_\infty$  denote the infinite cyclic cover of  $W'_0$ . Note that  $W'_2$  is a quotient of this covering space.  $\chi$  induces a  $C_q$ -covering of  $V_2$  and thus of  $W'_2$ . If we pull the  $C_q$ -covering of  $W'_2$  up to  $W'_\infty$ , we obtain  $\widetilde{W}'_\infty$ , a  $C_q \times C_\infty$ -covering of  $W'_2$ . If we identify properly  $F \times S^1$  in  $M'_2$ , this covering restricted to  $F \times S^1$  is given by

a character  $H_1(F \times S^1) \simeq H_1(F) \oplus H_1(S^1) \rightarrow C_q \times C_\infty$  that maps  $H_1(F)$  to zero in  $C_\infty$ ,  $H_1(S^1)$  to zero in  $C_q$  and isomorphically onto  $C_\infty$ . For this note: since  $\text{Hom}(H_1(F), \mathbb{Z}) = H^1(F) = [F, S^1]$ , we may define diffeomorphisms of  $F \times S^1$  that induce the identity on the second factor of  $H_1(F \times S^1) \approx H_1(F) \oplus \mathbb{Z}$ , and send  $(x, 0) \in H_1(F) \oplus \mathbb{Z}$ , to  $(x, f(x)) \in H_1(F) \oplus \mathbb{Z}$ , for any  $f \in \text{Hom}(H_1(F), \mathbb{Z})$ .

As in [Gil], choose inductively a collection of  $g$  disjoint curves in the kernel of  $\chi$  that form a metabolizer for the intersection form on  $H_1(F)/H_1(\partial F)$ . By taking a tubular neighborhood of these curves in  $F$ , we obtain a collection of  $S^1 \times I$  embedded in  $F$ . Using these embeddings we can attach round 2-handles  $(B^2 \times I) \times S^1$  along  $(S^1 \times I) \times S^1$  to the trivial cobordism  $M'_2 \times I$  and obtain a cobordism  $\Omega$  between  $M_2$  and  $M'_2$ .

Let  $U = W'_2 \cup_{M'_2} \Omega$  with boundary  $M_2$ . The  $C_q \times C_\infty$ -covering of  $W'_2$  extends uniquely to  $U$ . Note that  $\Omega$  may also be viewed as the result of attaching round 1-handles to  $M_2 \times I$ .

As in [Gil],  $\text{Sign}(W'_2) = \text{Sign}(V_2)$ . Since the intersection form on  $\Omega$  is zero, we get  $\text{Sign}(U) = \text{Sign}(W'_2) = \text{Sign}(V_2) = \sigma_L(-1)$ . The  $C_q \times C_\infty$ -covering of  $\Omega$ , restricted to each round 2-handle is  $q$  copies of  $B^2 \times I \times \mathbb{R}$  attached to the trivial cobordism  $\widetilde{M}'_\infty \times I$  along  $q$  copies of  $S^1 \times I \times \mathbb{R}$ . Using a Mayer-Vietoris sequence, one sees that the inclusion induces an isomorphism (which preserves the Hermitian form)

$$H_2^t(U; \mathbb{Q}(C_q)(t)) \simeq H_2^t(W'_2; \mathbb{Q}(C_q)(t)).$$

Thus, if  $w(W'_2)$  denotes the image of the intersection form on  $H_2^t(W'_2; \mathbb{Q}(C_q)(t))$  in  $\mathcal{W}(\mathbb{Q}(C_q)(t))$ , we get  $\sigma_1(\tau(M_2, \chi^+)) = \sigma_1(w(W'_2)) - \sigma_L(-1)$ .

If  $q$  is a prime power, we may apply Lemma 2 of [Gil] and conclude that  $H_i(\widetilde{W}'_\infty; \mathbb{Q})$  is finite dimensional for all  $i \neq 2$ . Thus,  $H_i^t(W'_2; \mathbb{Q}(C_q)(t))$  is zero for all  $i \neq 2$ . Since the Euler characteristic of  $W'_2$  with coefficients in  $\mathbb{Q}(C_q)(t)$  coincides with those with coefficients in  $\mathbb{Q}$ , we get  $\dim H_2^t(W'_2; \mathbb{Q}(C_q)(t)) = \chi(W'_2) = 2\chi(W'_0) = 2(1 - \chi(F)) = 2b_1(F)$ . Thus  $|\sigma_1(\tau(M_2, \chi^+) + \sigma_L(-1))| \leq 2b_1(F)$ . Hence,

$$\begin{aligned} |\sigma(L, \chi) + \sigma_L(-1)| &\leq |\sigma(L, \chi) - \sigma_1(\tau(M_2, \chi^+))| + |\sigma_1(\tau(M_2, \chi^+) + \sigma_L(-1))| \\ &\leq \eta(L, \chi) + \mu + 2(2g + \mu - 1) = \eta(L, \chi) + 4g + 3\mu - 2 \text{ by Theorem 3.4.} \quad \square \end{aligned}$$

## 5 Examples

Let  $L = L_1 \cup L_2$  be the link with two components of Figure 1 and  $S$  be the Seifert surface of  $L$  given by the picture. The squares with  $K$  denote two

parallel copies with linking number 0 of an arc tied in the knot  $K$ . Note that  $L$  is actually a family of examples. Specific links are determined by the choice of two parameters: a knot  $K$  and a positive integer  $h$ . Since  $S$  has genus  $h$ , the slice genus of  $L$  is at most  $h$ .

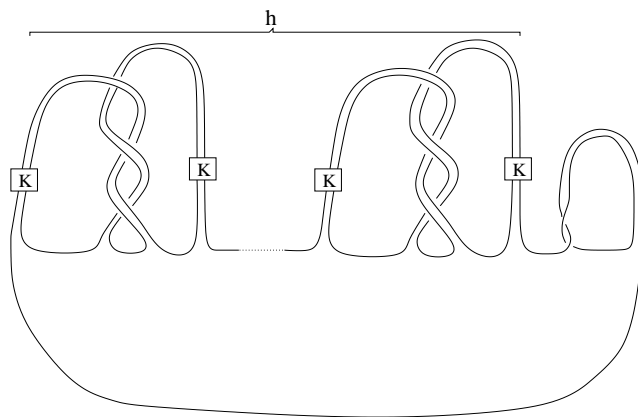


Figure 1: The link  $L$

One calculates that  $\sigma_L(\lambda) = 1$ , and  $n_L(\lambda) = 0$  for all  $\lambda$ . Thus, the Murasugi-Tristram inequality says nothing about the slice genus of  $L$ . In fact, if  $K$  is a slice knot, then one can surger this surface to obtain a smooth cylinder in the 4-ball with boundary  $L$ . Thus there can be no arguments based solely on a Seifert pairing for  $L$  that would imply that the slice genus is non-zero.

**Theorem 5.1** *If  $\sigma_K(e^{2i\pi/3}) \geq 2h$  or  $\sigma_K(e^{2i\pi/3}) \leq -2h - 2$ , then  $L$  has slice genus  $h$ .*

**Proof** Using [AK], a surgery presentation of  $N_2$  as surgery on a framed link of  $2h + 1$  components can be obtained from the surface  $S$  (see Figure 2).

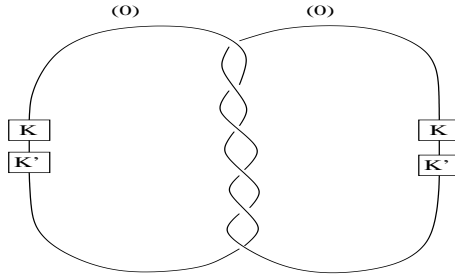
Let  $Q$  be the 3-manifold obtained from the link pictured in Figure 2. Here  $K'$  denotes  $K$  with the string orientation reversed. Since  $RP(3)$  is obtained by surgery on the unknot framed 2, we get:

$$N_2 = RP(3) \#_h Q.$$

The linking matrix of the framed link of the surgery presentation of  $N_2$  is

$$\Lambda = [2] \oplus \oplus^h \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}. \quad \Lambda \text{ is a presentation matrix of } (H_1(N_2)^*, \beta_L); \text{ we obtain}$$

$$H_1(N_2)^* \simeq \mathbb{Z}_2 \bigoplus \oplus^{2h} \mathbb{Z}_3$$

Figure 2: Surgery presentation of  $Q$ 

and  $\beta_L$  is given by the following matrix, with entries in  $\mathbb{Q}/\mathbb{Z}$ :

$$[1/2] \bigoplus \bigoplus^h \begin{bmatrix} 0 & 1/3 \\ 1/3 & 0 \end{bmatrix}.$$

By Theorem 4.1, if  $L$  bounds a surface of genus  $h - 1$  in  $B^4$ , then  $\beta_L$  must be decomposed as  $\beta_1 \oplus \beta_2$  where:

- 1)  $\beta_1$  has an even presentation matrix of rank  $2h - 1$ , and signature 1 (all we really need here is that it has a rank  $2h - 1$  presentation.)
- 2)  $\beta_2$  is metabolic and for all characters  $\chi$  of prime power order in some metabolizer of  $\beta_2$ , the following inequality holds:

$$(*) \quad |\sigma(L, \chi) + 1| - \eta(L, \chi) \leq 4h.$$

As  $\mathbb{Z}_2 \bigoplus \bigoplus^{2h} \mathbb{Z}_3$  does not have a rank  $2h - 1$  presentation,  $\beta_2$  is non-trivial. As metabolic forms are defined on groups whose cardinality is a square,  $\beta_2$  is defined on a group with no 2-torsion. Thus the metabolizer contains a non-trivial character of order three satisfying  $\beta_L(\chi, \chi) = 0$ .

The first homology of  $Q$  is  $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ , generated by, say,  $m_1$  and  $m_2$ , positive meridians of these components. Each of these components is oriented counter-clockwise. We first work out  $\sigma(Q, \chi)$  and  $\eta(Q, \chi)$  for characters of order three.

Let  $\chi_{(a_1, a_2)}$  denote the character on  $H_1(Q)$  sending  $m_j$  to  $e^{\frac{2i\pi a_j}{3}}$ , where the  $a_j$  take the values zero and  $\pm 1$ .

We use Proposition 2.4 to compute  $\sigma(Q, \chi_{(1,0)})$  and  $\eta(Q, \chi_{(1,0)})$  assuming that  $K$  is trivial. For this, one may adapt the trick illustrated on a link with 2 twists between the components [Gi2, Fig (3.3), Remark (3.65b)]. In the case  $K$  is the unknot, we obtain

$$\sigma(Q, \chi_{(1,0)}) = 1 \quad \text{and} \quad \eta(Q, \chi_{(1,0)}) = 0.$$

It is not difficult to see that inserting the knots of the type  $K$  changes the result as follows (note that  $K$  and  $K'$  have the same Tristram-Levine signatures):

$$\sigma(Q, \chi_{(1,0)}) = 1 + 2\sigma_K(e^{2\pi i/3}) \quad \text{and} \quad \eta(Q, \chi_{(1,0)}) = 0.$$

These same values hold for the characters  $\chi_{(-1,0)}$  and  $\chi_{(0,\pm 1)}$  by symmetry.

Using Proposition 2.4

$$\sigma(Q, \pm\chi_{(1,1)}) = -1 - 24/9 + 4\sigma_K(e^{2\pi i/3}), \quad \eta(Q, \pm\chi_{(1,1)}) = 0$$

$$\sigma(Q, \pm\chi_{(1,-1)}) = 4 + 24/9 + 4\sigma_K(e^{2\pi i/3}) \quad \text{and} \quad \eta(Q, \pm\chi_{(1,-1)}) = 1.$$

One also has

$$\sigma(Q, \chi_{(0,0)}) = 0 \quad \text{and} \quad \eta(Q, \chi_{(0,0)}) = 0.$$

Any order three character on  $N_2$  that is self annihilating under the linking form is given as the sum of the trivial character on  $RP(3)$  and characters of type  $\chi_{(0,0)}$ ,  $\chi_{(\pm 1,0)}$  and  $\chi_{(0,\pm 1)}$  on  $Q$  and characters of type  $\pm\chi_{(1,1)} + \pm\chi_{(1,-1)}$  on  $Q\#Q$ . Using Proposition 2.5, one can calculate  $\sigma(L, \chi)$  and  $\eta(L, \chi)$  for all these characters  $\chi$ . It is now a trivial matter to check that for every non-trivial character with  $\beta(\chi, \chi) = 0$ , the inequality (\*) is not satisfied.  $\square$

## References

- [AK] Akbulut, S., Kirby, R., *Branched covers of surfaces in 4-manifolds*, Math. Ann. **252**, 111-131 (1980).
- [AS] Atiyah, M. F., Singer, I. M., *The index of elliptic operators. III*, Ann. of Math. (2) **87**, 546-604 (1968).
- [CF] Conner, P. E., Floyd, E.E., *Differential Periodic Maps*, Ergebnisse der Mathematik und ihrer Grenzgebiete, **33**, Springer-Verlag, (1964).
- [CG1] Casson, A. J., Gordon, C. Mc A., *Cobordism of classical knots*, Progr. Math., **62**, A La Recherche de la Topologie Perdue, Birkhauser, Boston, MA, 181-199 (1986).
- [CG2] Casson, A. J., Gordon, C. Mc A., *On slice knots in dimension three*, Proc. Symp. in Pure Math. XXX, **2**, 39-53 (1978).
- [Gi1] Gilmer, P. M., *On the slice genus of knots*, Invent. Math. **66**, 191-197 (1982).
- [Gi2] Gilmer, P. M., *Configurations of surfaces in 4-manifolds*, Trans. Amer. Math. Soc. **264**, 353-380 (1981).
- [Gi3] Gilmer, P. M., *Slice knots in  $S^3$* , Quart. J. Math. Oxford **34**, 305-322 (1983).
- [Gi4] Gilmer, P. M., *Classical knot and link concordance*, Comment. Math. Helv. **68**, 1-19 (1993).

- [Gi5] Gilmer, P. M., *Signatures of singular branched covers*, Math. Ann. **295** (4), 643–659 (1993).
- [GL] Gilmer, P. M., Livingston, C., *The Casson-Gordon invariant and link concordance*, Topology **31**, (3), 475–492 (1992).
- [G] Gordon, C. McA., *Some aspects of classical knot theory*, Knot theory (Proc. Sem., Plans-sur-Bex, 1977), Lecture Notes in Math., **685**, Springer Verlag, Berlin, 1-60 (1978).
- [K] Kirby, R. C., *The Topology of 4-manifolds*, Lecture Notes in Math **1374** Springer Verlag, Berlin (1989).
- [Le] Levine, J., *Knot cobordism groups in codimension two*, Comment. Math. Helv. **44** 229–244 (1969).
- [L] Lines D., *Cobordisme de noeuds fibrés et de leur monodromie*, Knots, braids and singularities (Plans-sur-Bex, 1982), 147–173, Monogr. Enseign. Math., **31**, Enseignement Math., Geneva, (1983).
- [Li] Litherland, R. A., *Cobordism of satellite knots*, Four-Manifold Theory (Durham, N.H., 1982), Contemp. Math., **35**, Amer. Math. Soc., Providence, RI, 327–362 (1984).
- [M] Murasugi, K., *On a certain numerical invariant of link types*, Trans. Amer. Math. Soc. **117**, 387–422 (1965).
- [N] Naik, S. *Casson-Gordon invariants of genus one knots and concordance to reverses*, J. Knot Theory Ramifications **5**, 661–677 (1996).
- [T] Tristram, A. G., *Some cobordism invariants for links*, Proc. Camb. Philos. Soc., **66**, 251-264 (1969).
- [V] Viro, O. Ja. *Branched coverings of manifolds with boundary, and invariants of links. I*, Math. USSR-Izv. **7** 1239–1256 (1973).

Laboratoire I.R.M.A. Université Louis Pasteur  
Strasbourg, France

and

Department of Mathematics, Louisiana State University  
Baton Rouge, LA 70803, USA

Email: vincent.florens@irma.u-strasbg.fr and gilmer@math.lsu.edu